

Chapter 1 Ordinary Differential Equations

The solution of the differential equation $\frac{dy}{dx} = f(x)$ is

$$y(x) = \int f(x)dx + c \equiv y_p(x) + y_h(x)$$

where $y_p(x)$ and $y_h(x)$ are particular solution and homogeneous solution, respectively.

Ex. $\frac{dy}{dx} = \sin(ax) + \cos(bx) + x^n + e^x + x^{-1}$

$$\begin{aligned} y &= \int [\sin(ax) + \cos(bx) + x^n + e^x + x^{-1}]dx + c \\ &= -\frac{1}{a}\cos(ax) + \frac{1}{b}\sin(bx) + \frac{x^{n+1}}{n+1} + e^x + \ln x + c \end{aligned}$$

Ex. $\frac{dy}{dx} = \frac{1}{x} + 2x \cos(ax)$

$$\begin{aligned} y &= \int \left[\frac{1}{x} + 2x \cos(ax) \right] dx + c = \int \frac{1}{x} dx + \int 2x \cos(ax) dx + c \\ &= \ln x + \frac{1}{a} \int 2x d \sin(ax) + c = \ln x + \frac{1}{a} [2x \sin(ax) - 2 \int \sin(ax) dx] + c \\ &= \ln x + \frac{2}{a} x \sin(ax) - \frac{2}{a} \left(-\frac{1}{a} \right) \int d \cos(ax) + c \\ &= \ln x + \frac{2}{a} x \sin(ax) + \frac{2}{a^2} \cos(ax) + c \end{aligned}$$

Ex. $\frac{d^2 y}{dx^2} = \frac{1}{x} + 2x \cos(ax)$

$$\begin{aligned} \frac{dy}{dx} &= \ln x + \frac{2}{a} x \sin(ax) + \frac{2}{a^2} \cos(ax) + c_1 \\ y &= \int \left[\ln x + \frac{2}{a} x \sin(ax) + \frac{2}{a^2} \cos(ax) \right] dx + c_1 x + c_2 \\ &= \int \ln x dx + \frac{2}{a} \left(-\frac{1}{a} \right) \int x d \cos(ax) + \frac{2}{a^2} \int \cos(ax) dx + c_1 x + c_2 \\ &= (x \ln x - \int x d \ln x) - \frac{2}{a^2} [x \cos(ax) - \int \cos(ax) dx] + \frac{2}{a^3} \sin(ax) + c_1 x + c_2 \\ &= (x \ln x - \int x x^{-1} dx) - \frac{2}{a^2} [x \cos(ax) - \frac{1}{a} \sin(ax)] + \frac{2}{a^3} \sin(ax) + c_1 x + c_2 \\ &= (x \ln x - x) - \frac{2}{a^2} x \cos(ax) + \frac{4}{a^3} \sin(ax) + c_1 x + c_2 \end{aligned}$$

$$= x \ln x - \frac{2}{a^2} x \cos(ax) + \frac{4}{a^3} \sin(ax) + (c_1 - 1)x + c_2$$

Consider the equation $f(x, y) = c$

$$\text{Then } 0 = df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \equiv M(x, y)dx + N(x, y)dy$$

§ Exact D. E.

Definition.

The equation $M(x, y)dx + N(x, y)dy = 0$ is an exact differential equation if

and only if there is a function $f(x, y)$ such that

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$$

Theorem

$$M(x, y)dx + N(x, y)dy = 0 \text{ is exact if and only if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Ex. Solve $(y^3 + 2x)dx + (3xy^2 + 1)dy = 0$

$$M(x, y) = y^3 + 2x, \quad N(x, y) = 3xy^2 + 1, \quad \frac{\partial M}{\partial y} = 3y^2, \quad \frac{\partial N}{\partial x} = 3y^2, \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

$(y^3 + 2x)dx + (3xy^2 + 1)dy = 0$ is an exact differential equation

$$\frac{\partial f}{\partial x} = M(x, y) = y^3 + 2x \text{ implies } f(x, y) = xy^3 + x^2 + g(y)$$

$$\frac{\partial f}{\partial y} = 3xy^2 + \frac{dg}{dy} = N(x, y) = 3xy^2 + 1 \text{ implies } g(y) = y + c$$

So, $f(x, y) = xy^3 + x^2 + y + c$.

Another way, $y^3 dx + dx^2 + xdy^3 + dy = 0$ or $(y^3 dx + xdy^3) + dx^2 + dy = 0$

$$xy^3 + x^2 + y = c$$

Ex. Solve $(x^3 + \cos y + \frac{1}{x})dy = (\frac{y}{x^2} - 3x^2 y)dx$

$$x^3 dy + \cos y dy + \frac{1}{x} dy + 3x^2 y dx - \frac{y}{x^2} dx = 0, \quad x^3 dy + \cos y dy + \frac{1}{x} dy + y dx^3 + y d\frac{1}{x} = 0$$

$$x^3 dy + y dx^3 + \cos y dy + y d\frac{1}{x} + \frac{1}{x} dy = 0, \quad d(x^3 y) + \cos y dy + d\left(\frac{y}{x}\right) = 0$$

$$x^3 y + \sin y + \frac{y}{x} = c$$

Ex. Solve $(2xy + x^3)dx + (x^2 + y^2)dy = 0$

$$0 = (2xy + x^3)dx + (x^2 + y^2)dy = ydx^2 + \frac{1}{4}dx^4 + x^2 dy + \frac{1}{3}dy^3$$

$$0 = (ydx^2 + x^2 dy) + \frac{1}{4}dx^4 + \frac{1}{3}dy^3, \quad x^2 y + \frac{1}{4}x^4 + \frac{1}{3}y^3 = c.$$

§ Integrating factor

Suppose $M(x, y)dx + N(x, y)dy = 0$ not to be an exact differential equation.

If there exists an $\mu(x, y)$ such that $\mu M(x, y)dx + \mu N(x, y)dy = 0$ to be an exact differential equation, then $\mu(x, y)$ is called an integrating factor.

Ex. $(x^2 + y^2 - x)dx - ydy = 0$, $M(x, y) = x^2 + y^2 - x$, $N(x, y) = -y$,

$$\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = 0. \text{ However,}$$

$$(x^2 + y^2)dx - (xdx + ydy) = (x^2 + y^2)dx - \frac{1}{2}d(x^2 + y^2) = 0$$

$$\mu(x, y) = \frac{1}{x^2 + y^2}, \quad 2dx - \frac{1}{x^2 + y^2}d(x^2 + y^2) = 0, \quad 2x - \ln(x^2 + y^2) = c$$

Ex. $ydx + (x^2 y^3 + x)dy = 0$ is not an exact differential equation.

$$x^2 y^3 dy + (xdy + ydx) = x^2 y^3 dy + d(xy) = 0$$

$$\mu(x, y) = \frac{1}{x^2 y^2}, \quad ydy + \frac{1}{x^2 y^2}d(xy) = 0, \quad \frac{1}{2}y^2 - \frac{1}{xy} = c$$

Ex. $xdy - ydx = (4x^2 + y^2)dy$ is not an exact differential equation.

$$\frac{1}{x^2}(xdy - ydx) = \frac{1}{x^2}(4x^2 + y^2)dy \rightarrow d\left(\frac{y}{x}\right) = \left[4 + \left(\frac{y}{x}\right)^2\right]dy$$

Let $u = y/x \rightarrow \frac{du}{4 + u^2} = dy = \frac{2}{4} \frac{d(u/2)}{1 + (u/2)^2} = \frac{1}{2} \frac{d(u/2)}{1 + (u/2)^2},$

Let $u/2 = \tan \theta \rightarrow d(u/2) = \sec^2 \theta d\theta \rightarrow$

$$dy = \frac{1}{2} d\theta \rightarrow 2y = \theta + c = \tan^{-1}(u/2) + c = \tan^{-1}(y/2x) + c$$

$$\rightarrow y/2x = \tan(2y - c).$$

Ex. $y(1+xy)dx + (2y-x)dy = 0$

$$ydx - xdy + \frac{1}{2}y^2dx^2 + dy^2 = 0 \rightarrow \frac{1}{y^2}(ydx - xdy) + d(\frac{1}{2}x^2) + \frac{1}{y^2}dy^2 = 0$$

$$\rightarrow d(\frac{x}{y}) + d(\frac{1}{2}x^2) + d\ln y^2 = 0 \rightarrow \frac{x}{y} + \frac{1}{2}x^2 + \ln y^2 = c$$

Ex. $2x \ln y dx + \frac{1+x^2}{y} dy = 0$

$$\frac{2x}{1+x^2} dx + \frac{1}{y \ln y} dy = 0 \rightarrow \frac{d(x^2+1)}{1+x^2} + \frac{1}{\ln y} d\ln y = 0 \rightarrow \ln(x^2+1) + \ln(\ln y) = c$$

$$(x^2+1)\ln y = e^c$$

§ Separation of variables

$$f(x)dx + g(y)dy = 0 \rightarrow \int f(x)dx + \int g(y)dy = c$$

Ex. $\frac{dy}{dx} + 4x + \sin x = 6y^2 \cos y^3 \frac{dy}{dx} - \frac{4}{1+x^2}$

$$dy - 6y^2 \cos y^3 dy = (-4x - \sin x - \frac{4}{1+x^2})dx \rightarrow$$

$$dy - 2 \cos y^3 dy^3 = -2dx^2 + d \cos x - 4d \tan^{-1} x \rightarrow$$

$$y - 2 \sin y^3 + 2x^2 - \cos x + 4 \tan^{-1} x = c$$

Ex. $(2x \cos y)dx + x^2(\sec y - \sin y)dy = 0 \rightarrow \frac{2}{x}dx + \frac{\sec y - \sin y}{\cos y}dy = 0 \rightarrow$

$$2d \ln x + d \tan y + d \ln \cos y = 0 \rightarrow \ln x^2 + \tan y + \ln \cos y = c$$

§ Homogeneous Eq.

Given a function $f(x_1, x_2, \dots, x_m)$ if $f(\lambda x_1, \lambda x_2, \dots, \lambda x_m) = \lambda^n f(x_1, x_2, \dots, x_m)$

Then $f(x_1, x_2, \dots, x_m)$ is homogeneous of order n.

Ex. Is $f(x, y) = x[\ln \sqrt{x^2 + y^2} - \ln y] + ye^{x/y}$ homogeneous

$$f(x, y) = x \ln(\sqrt{x^2 + y^2} / y) + ye^{x/y} \rightarrow$$

$$f(\lambda x, \lambda y) = \lambda x \ln(\lambda \sqrt{x^2 + y^2} / \lambda y) + \lambda ye^{\lambda x / \lambda y} \rightarrow f(\lambda x, \lambda y) = \lambda f(x, y)$$

$$\text{If } M(\lambda x, \lambda y) = \lambda M(x, y) \text{ and } N(\lambda x, \lambda y) = \lambda N(x, y)$$

$$\text{Then } M(x, y)dx + N(x, y)dy = 0 \rightarrow \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} = -\frac{M(1, y/x)}{N(1, y/x)} \triangleq R(y/x)$$

$$\text{Let } u = y/x \rightarrow \frac{dy}{dx} = x \frac{du}{dx} + u = R(u) \rightarrow \int \frac{du}{R(u) - u} = \int \frac{dx}{x} = \ln x$$

$$\text{Ex. } (x^2 + 3y^2)dx - 2xydy = 0 \text{ is homogeneous}$$

$$y = xu \rightarrow (1 + 3u^2)dx - 2u(xdu + udx) = 0 \rightarrow (1 + u^2)dx - xdu^2 = 0 \rightarrow \frac{dx}{x} - \frac{du^2}{1 + u^2} = 0$$

$$\rightarrow \ln x - \ln(1 + u^2) = c \rightarrow x / (1 + u^2) = e^c \rightarrow x^3 / (x^2 + y^2) = e^c$$

$$\text{Ex. } (x^2 + y^2)dx = 2xydy$$

$$\text{Ex. } x \frac{dy}{dx} - y = (x^2 - y^2)^{1/2} \rightarrow [(x^2 - y^2)^{1/2} + y]dx - xdy = 0 \text{ is homogenous}$$

$$\text{Let } y = xu \rightarrow x[(1 - u^2)^{1/2} + u]dx - x(udx + xdu) = 0 \rightarrow \frac{dx}{x} - \frac{du}{(1 - u^2)^{1/2}} = 0$$

$$\text{Let } u = \sin v \rightarrow \frac{dx}{x} - dv = 0 \rightarrow \ln x - v = c \rightarrow \ln x - \sin^{-1} u = c \rightarrow$$

$$\ln x - \sin^{-1}(y/x) = c$$

$$\text{Ex. } x \frac{dy}{dx} = y + (x^2 + y^2)^{1/2} \text{ is homogeneous}$$

$$\text{Let } y = xu \rightarrow x(xdu + udx) = x[u + (1 + u^2)^{1/2}]dx \rightarrow \frac{du}{(1 + u^2)^{1/2}} = \frac{dx}{x}$$

$$\text{Let } u = \sinh \theta \rightarrow \frac{\cosh \theta d\theta}{\cosh \theta} = \frac{dx}{x} \rightarrow \theta - \ln x = c \rightarrow \sinh^{-1} u - \ln x = c \rightarrow$$

$$\sinh^{-1}(y/x) - \ln x = c$$

$$\text{Ex. } \frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x}$$

$$\text{Let } y = xu \rightarrow u + x \frac{du}{dx} = u^2 + u \rightarrow \frac{du}{u^2} = \frac{dx}{x} \rightarrow \ln x + u^{-1} = c \rightarrow \ln x + (x/y) = c$$

$$\text{Consider } \frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

Case 1. $aB \neq Ab$.

Let (x_o, y_o) be the intersection of $ax + by + c = 0$ and $Ax + by + C = 0$.

Let $x = u + x_o$, $y = v + y_o$, then $\frac{dv}{du} = \frac{au + bv}{Au + Bv}$

Case 2. $aB = Ab$

Let $v = ax + by$ if $a \neq 0$ or $v = Ax + By$ if $a = 0$.

Ex. $\frac{dy}{dx} = \frac{x - y + 5}{x + y - 1}$

$(-2, 3)$ is the intersection of $x - y + 5 = 0$ and $x + y - 1 = 0$

Let $x = u - 2$, $y = v + 3$, then $\frac{dv}{du} = \frac{u - v}{u + v}$ or $(u + v)dv = (u - v)du$

$$\rightarrow u dv + v du + v dv - u du = 0 \rightarrow d(uv) + d\left(\frac{1}{2}v^2\right) - d\left(\frac{1}{2}u^2\right) = 0$$

$$\rightarrow uv + \frac{1}{2}v^2 - \frac{1}{2}u^2 = c \rightarrow (x + 2)(y - 3) + \frac{1}{2}(y - 3)^2 - \frac{1}{2}(x + 2)^2 = c.$$

Ex. $\frac{dy}{dx} = \frac{4x - 2y + 5}{2x - y}$

Let $v = 2x - y \rightarrow dy = 2dx - dv$, $2 - \frac{dv}{dx} = \frac{2v + 5}{v} \rightarrow -v dv = 5dx \rightarrow 5x + \frac{1}{2}v^2 = c$

$$\rightarrow 5x + \frac{1}{2}(2x - y)^2 = c$$

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$$\rightarrow u dv + v du + v dv - u du = 0 \rightarrow d(uv) + d\left(\frac{1}{2}v^2\right) - d\left(\frac{1}{2}u^2\right) = 0$$

$$\rightarrow uv + \frac{1}{2}v^2 - \frac{1}{2}u^2 = c \rightarrow (x+2)(y-3) + \frac{1}{2}(y-3)^2 - \frac{1}{2}(x+2)^2 = c.$$

Ex. $\frac{dy}{dx} = \frac{4x-2y+5}{2x-y}$

Let $v = 2x - y \rightarrow dy = 2dx - dv$, $2 - \frac{dv}{dx} = \frac{2v+5}{v} \rightarrow -v dv = 5dx \rightarrow 5x + \frac{1}{2}v^2 = c$

$$\rightarrow 5x + \frac{1}{2}(2x-y)^2 = c$$

If $f(x, y)$ is a homogeneous function of degree n , then $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$

Pf: $f(\lambda x, \lambda y) = \lambda^n f(x, y)$,

$$\frac{d}{d\lambda} f(\lambda x, \lambda y) = \frac{\partial f}{\partial(\lambda x)} \frac{\partial(\lambda x)}{\partial \lambda} + \frac{\partial f}{\partial(\lambda y)} \frac{\partial(\lambda y)}{\partial \lambda} = n \lambda^{n-1} f(x, y)$$

$$x \frac{\partial f}{\partial(\lambda x)} + y \frac{\partial f}{\partial(\lambda y)} = n \lambda^{n-1} f(x, y)$$

Let $\lambda = 1$, then $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M dx + N dy$$

Properties:

a. If $M(x, y)dx + N(x, y)dy = 0$ is homogeneous and exact, then its

solution is

$$xM(x, y) + yN(x, y) = c$$

b. If $M(x, y)dx + N(x, y)dy = 0$ is homogeneous, then

$$\frac{1}{xM + yN} \text{ is an integrating factor}$$

§ Integrating factor

$$F(x) \frac{dy}{dx} + G(x)y = H(x) \rightarrow \frac{dy}{dx} + \frac{G(x)}{F(x)}y = \frac{H(x)}{F(x)}$$

Integrating factor $\mu(x) = \exp\left(\int \frac{G}{F} dx\right),$

$$d\mu(x) = \exp\left(\int \frac{G}{F} dx\right) \frac{G}{F} dx = \mu(x) \frac{G}{F} dx$$

$$\mu(x) \frac{dy}{dx} + \mu(x) \frac{G(x)}{F(x)} y = \mu(x) \frac{H(x)}{F(x)} \rightarrow \mu(x) \frac{dy}{dx} + \mu(x) \frac{G(x)}{F(x)} y = \mu(x) \frac{H(x)}{F(x)}$$

$$\rightarrow \mu(x) dy + y \mu(x) \frac{G(x)}{F(x)} dx = \mu(x) \frac{H(x)}{F(x)} dx \rightarrow d[\mu(x)y] = \mu(x) \frac{H(x)}{F(x)} dx$$

$$\rightarrow y = \frac{1}{\mu(x)} \left(\int \mu(x) \frac{H(x)}{F(x)} dx + c \right)$$

Ex. $(2y + x^2)dx = xdy \rightarrow \frac{dy}{dx} + \left(\frac{-2}{x}\right)y = x$

$$\mu(x) = \exp\left(-2 \int \frac{dx}{x}\right) = \exp(-2 \ln x) = \exp(\ln x^{-2}) = x^{-2}$$

$$x^{-2} \frac{dy}{dx} - 2x^{-3}y = x^{-1} \rightarrow d(x^{-2}y) = x^{-1}dx \rightarrow x^{-2}y - \ln x = c \rightarrow y = x^2(c + \ln x),$$

Ex. $\frac{dy}{dx} + 2xy + x = e^{-x^2} \rightarrow \frac{dy}{dx} + 2xy = e^{-x^2} - x$

$$\mu(x) = \exp\left(2 \int x dx\right) = \exp(x^2) = e^{x^2}, \quad e^{x^2} \frac{dy}{dx} + 2xe^{x^2}y = 1 - xe^{x^2}$$

$$d(e^{x^2}y) = (1 - xe^{x^2})dx \rightarrow e^{x^2}y = \int (1 - xe^{x^2})dx = x - \frac{1}{2} \int e^{x^2} dx^2 = x - e^{x^2} + c$$

Ex. $\sin x \frac{dy}{dx} + y \cos x = \cos 2x$

Ex. $(5x^2 + 1) \frac{dy}{dx} - 20xy = 10x$

$$\frac{dy}{dx} - 2 \frac{10x}{5x^2 + 1} xy = \frac{10x}{5x^2 + 1},$$

$$\mu(x) = \exp\left(-2 \int \frac{10x}{5x^2 + 1} dx\right) = \exp[-2 \ln(5x^2 + 1)] = \frac{1}{(5x^2 + 1)^2},$$

$$\frac{1}{(5x^2 + 1)^2} \frac{dy}{dx} - 2 \frac{10x}{(5x^2 + 1)^3} xy = \frac{10x}{(5x^2 + 1)^3}$$

$$\frac{y}{(5x^2 + 1)^2} = \int \frac{10x}{(5x^2 + 1)^3} dx = \int \frac{d(5x^2 + 1)}{(5x^2 + 1)^3} = -\frac{1}{2} (5x^2 + 1)^{-2} + c$$

$$y = -\frac{1}{2} + c(5x^2 + 1)^2$$

§ Special 1st order eq.

1. Bernoulli eq. $\frac{dy}{dx} + p(x)y = Q(x)y^n$

$$y^{-n} \frac{dy}{dx} + p(x)y^{1-n} = Q(x) \rightarrow \frac{1}{1-n} \frac{dy^{1-n}}{dx} + p(x)y^{1-n} = Q(x)$$

Let $Y = y^{1-n} \rightarrow \frac{dY}{dx} + (1-n)p(x)Y = (1-n)Q(x)$

$$\mu(x) = \exp\left[\int (1-n)p(x)dx\right] \rightarrow \mu(x) \frac{dY}{dx} + (1-n)p(x)\mu(x)Y = (1-n)\mu(x)Q(x)$$

$$\rightarrow \frac{d}{dx}[\mu(x)Y] = (1-n)\mu(x)Q(x) \rightarrow Y(x) = \frac{1}{\mu(x)}\left[\int (1-n)\mu(x)Q(x)dx + c\right]$$

$$y = Y^{\frac{1}{1-n}}$$

Ex. $3x \frac{dy}{dx} + y + x^2 y^4 = 0 \rightarrow 3xy^{-4} \frac{dy}{dx} + y^{-3} + x^2 = 0 \rightarrow -x \frac{dy^{-3}}{dx} + y^{-3} + x^2 = 0$

$$\rightarrow \frac{dy^{-3}}{dx} - \frac{1}{x} y^{-3} - x = 0, \quad \mu(x) = \exp\left(-\int \frac{dx}{x}\right) = \frac{1}{x}$$

$$\rightarrow \frac{1}{x} \frac{dy^{-3}}{dx} - \frac{1}{x^2} y^{-3} - 1 = 0 \rightarrow \frac{d}{dx}\left(\frac{y^{-3}}{x}\right) = 1 \rightarrow y^{-3} = x(x+c) \rightarrow y = \left[\frac{1}{x(x+c)}\right]^{1/3}.$$

2. Riccati Eq. $\frac{dy}{dx} = p(x)y^2 + Q(x)y + R(x)$

If $y_1(x)$ is one solution then $\frac{dy_1}{dx} = p(x)y_1^2 + Q(x)y_1 + R(x)$

$$\frac{d(y-y_1)}{dx} = p(x)(y^2 - y_1^2) + Q(x)(y - y_1)$$

Let $y - y_1 = z, \rightarrow$

$$\frac{dz}{dx} = P(x)(z^2 + 2zy_1) + Q(x)z = P(x)z^2 + (2y_1P + Q)z \dots \text{Bernoulli Eq.}$$

Ex. $\frac{dy}{dx} = y^2 + (1-2x)y + (x^2 - x + 1)$

Try $y_1 = ax + b \rightarrow a = (ax+b)^2 + (1-2x)(ax+b) + (x^2 - x + 1)$

$$\rightarrow a = 1, \quad b = 0.$$

Let $y = x + z \rightarrow \frac{dz}{dx} = z^2 + z \rightarrow \frac{dz}{z} - \frac{dz}{z+1} = dx \rightarrow \ln \frac{z}{z+1} = x + c$

$$\rightarrow \frac{z}{z+1} = e^x e^c \rightarrow \frac{z+1}{z} = e^{-x} e^{-c} = c_1 e^{-x} \rightarrow \frac{1}{z} = c_1 e^{-x} - 1 \rightarrow z = (c_1 e^{-x} - 1)^{-1}$$

$$y = (c_1 e^{-x} - 1)^{-1} + x$$

Ex. $\frac{dy}{dx} = (x+y)(x+y-2)$

Try $y_1 = ax + b \rightarrow a = (x+ax+b)(x+ax+b-2) \rightarrow y_1 = -x+1$

Let $y = (-x+1) + z \rightarrow -1 + \frac{dz}{dx} = z^2 - 1 \rightarrow -z^{-1} = x+c \rightarrow z = -(x+c)^{-1}$

$y = 1 - x - (x+c)^{-1}$

Ex. $\frac{dy}{dx} = (y-1)(\frac{1}{x} + y)$

Let $y = 1 + z \rightarrow \frac{dz}{dx} = z(\frac{1}{x} + 1 + z) = z^2 + z(1 + \frac{1}{x}) \rightarrow z^{-2} \frac{dz}{dx} - (1 + \frac{1}{x})z^{-1} = 1$

$\rightarrow \frac{dz^{-1}}{dx} + (1 + \frac{1}{x})z^{-1} = -1, \quad \mu(x) = \exp[\int (1 + \frac{1}{x})dx] = xe^x$

$\frac{d}{dx}(xe^x z^{-1}) = -xe^x \rightarrow xe^x z^{-1} = (1-x)e^x + c \rightarrow z = \frac{xe^x}{(1-x)e^x + c}$

$y = 1 + \frac{xe^x}{(1-x)e^x + c}.$

3. Clairaut Eq. $y = x \frac{dy}{dx} + f(\frac{dy}{dx})$

The solution is $y = mx + f(m).$

Ex. $y = x \frac{dy}{dx} - 4(\frac{dy}{dx})^3, \quad y = x \frac{dy}{dx} + \frac{1}{1 + \frac{dy}{dx}}, \quad y = x \frac{dy}{dx} - \frac{1}{4}(\frac{dy}{dx})^2$

§ High order L. D. E.

The operator $L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_0y$ is linear, i.e.,

$L(\alpha y_1 + \beta y_2) = \alpha L(y_1) + \beta L(y_2)$

The differential equation $L(y) = 0$ has n homogeneous solutions $y_1, \dots,$

y_n

The combination $y = \sum_{i=1}^n c_i y_i$

Consider the equation $a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_0 y = 0$

Usually, the solution y can be set as the form $y = e^{\lambda x}$, then the

eigenvalue λ are determined by the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0$$

If all λ_i are distinct, then $y = \sum_{i=1}^n c_i e^{\lambda_i x}$.

§ Reduction of order

If $u(x)$ is one solution for $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$

Then $y(x) = u(x)v(x)$, where $v(x)$ will be determined

$$y'(x) = u'(x)v(x) + u(x)v'(x), \quad y''(x) = u''(x)v(x) + u(x)v''(x) + 2u'(x)v'(x)$$

$$a_0[u''v + uv'' + 2u'v'] + a_1(u'v + uv') + a_2uv = 0$$

$$a_0uv'' + (2a_0u' + a_1u)v' + (a_0u'' + a_1u' + a_2u)v = 0$$

$$\text{Let } z = v' \rightarrow a_0uz' + (2a_0u' + a_1u)z = 0 \rightarrow z' + \frac{2a_0u' + a_1u}{a_0u}z = 0$$

$$\frac{dz}{z} = -\frac{2a_0u' + a_1u}{a_0u}dx \rightarrow z = \exp\left[-\int \frac{2a_0u' + a_1u}{a_0u}dx + c\right] \rightarrow$$

$$v = \int \exp\left[-\int \frac{2a_0u' + a_1u}{a_0u}dx + c\right]dx + b$$

§ Variation of parameters

Let $u(x)$ and $v(x)$ be two solution of the equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

Then the solution of the equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x)$

can be expressed as $y(x) = y_1u + y_2v$, where y_1 and y_2 will be determined.

Let $y_1' u + y_2' v = 0$, then $y'(x) = y_1u' + y_2v'$, $y''(x) = y_1'u' + y_2'v' + y_1u'' + y_2v''$,

$$a_0(y_1'u' + y_2'v' + y_1u'' + y_2v'') + a_1(y_1u' + y_2v') + a_2(y_1u + y_2v) = f$$

$$a_0(y_1'u' + y_2'v') + y_1(a_0u'' + a_1u' + a_2u) + y_2(a_0v'' + a_1v' + a_2v) = f$$

$a_0(y_1' u' + y_2' v') = f$. Both y_1 and y_2 are determined by the following two equations: $y_1' u + y_2' v = 0$ and $a_0(y_1' u' + y_2' v') = f$.

Ex. $y'' + 3y' + 2y = 0$

Let $y = e^{\lambda x} \rightarrow \lambda^2 + 3\lambda + 2 = 0, \lambda = -1, -2$

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Ex. $y'' - 2y' + y = 0$

Let $y = e^{\lambda x}, \lambda^2 - 2\lambda + 1 = 0, \lambda = 1, 1. y = e^x$ is one solution.

Let $y = u(x)e^x$, then $y' = u'(x)e^x + u(x)e^x, y'' = u''(x)e^x + 2u'(x)e^x + u(x)e^x$

$$y'' - 2y' + y = u''(x)e^x = 0, u(x) = c_1 x + c_2, y(x) = (c_1 x + c_2)e^x$$

Ex. $y'' + 2y' + y = 0$

Let $y = e^{\lambda x} \rightarrow \lambda^2 + 2\lambda + 1 = 0, \lambda = -1, -1 \rightarrow y_1 = e^{-x},$

Let $y = y_1 u \rightarrow y_1 u'' + 2(y_1' + y_1)u' + (y_1'' + 2y_1' + y_1)u = 0 \rightarrow u'' = 0$

$$u = c_1 x + c_2 \rightarrow y = e^{-x}(c_1 x + c_2)$$

Ex. $y''' + 3y'' + 3y' + y = 0$

Let $y = e^{\lambda x} \rightarrow \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0, \lambda = -1, -1, -1 \rightarrow y_1 = e^{-x},$

Let $y = y_1 u \rightarrow y = e^{-x} u, y' = e^{-x}(u' - u), y'' = e^{-x}(u'' - 2u' + u),$

$$y''' = e^{-x}(u''' - 3u'' + 3u' - u) \rightarrow u''' = 0 \rightarrow u = c_1 x^2 + c_2 x + c_3$$

$$y = e^{-x}(c_1 x^2 + c_2 x + c_3).$$

Ex. $y'' + y' + y = 0 \rightarrow \lambda^2 + \lambda + 1 = 0, \lambda = -\frac{1}{2} \pm j \frac{\sqrt{3}}{2}$

$$\begin{aligned}
y &= c_1 e^{(-\frac{1}{2} + j\frac{\sqrt{3}}{2})x} + c_2 e^{(-\frac{1}{2} - j\frac{\sqrt{3}}{2})x} = e^{-\frac{1}{2}x} [c_1 e^{j\frac{\sqrt{3}}{2}x} + c_2 e^{-j\frac{\sqrt{3}}{2}x}] \\
&= e^{-\frac{1}{2}x} \{c_1 [\cos(\frac{\sqrt{3}}{2}x) + j \sin(\frac{\sqrt{3}}{2}x)] + c_2 [\cos(\frac{\sqrt{3}}{2}x) - j \sin(\frac{\sqrt{3}}{2}x)]\} \\
&= e^{-\frac{1}{2}x} \{A_1 \cos(\frac{\sqrt{3}}{2}x) + B_2 \sin(\frac{\sqrt{3}}{2}x)\}
\end{aligned}$$

Ex. $y^{(8)} + 6y^{(6)} - 32y^{(2)} = 0$

Let $y = e^{\lambda x} \rightarrow (\lambda^2 + 4)^2 \lambda^2 (\lambda^2 - 2) = 0 \rightarrow \lambda = 0, 0, \pm\sqrt{2}, 2j, 2j, -2j, -2j$

$$y = c_1 + c_2 x + c_3 e^{\sqrt{2}x} + c_4 e^{-\sqrt{2}x} + c_5 x e^{j2x} + c_6 e^{j2x} + c_7 x e^{-j2x} + c_8 e^{-j2x}$$

$$y = c_1 + c_2 x + c_3 e^{\sqrt{2}x} + c_4 e^{-\sqrt{2}x} + x[d_5 \cos 2x + d_6 \sin 2x] + [d_7 \cos 2x + d_8 \sin 2x]$$

§. Nonhomogeneous solution

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(x)$$

$$y = y_h + y_p$$

Ex. $y' + y = e^{-x}$

Method 1 $e^x (y' + y) = (e^x y)' = e^x e^{-x} = 1, \quad e^x y = x + c, \quad y = e^{-x} (x + c)$

Method 2 $(D+1)(D+1)y = (D+1)e^{-x} = 0$, where $D \equiv \frac{d}{dx}$

$$y = c_1 e^{-x} + c_2 x e^{-x}, \quad y' + y = c_2 e^{-x} = e^{-x}, \quad c_2 = 1, \quad y = c_1 e^{-x} + x e^{-x}$$

Ex. $y'' + 4y' + 3y = 5e^{2x}$

Method 1. $y_h = c_1 e^{-x} + c_2 e^{-3x}, \quad y_p = c_3 e^{2x} \rightarrow (4+8+3)c_3 = 5$

$$y = c_1 e^{-x} + c_2 e^{-3x} + \frac{1}{3} e^{2x}$$

Method 2. $(D-2)e^{2x} = 0 \rightarrow (D-2)(y'' + 4y' + 3y) = 5(D-2)e^{2x} = 0$

$$y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{2x} \rightarrow y'' + 4y' + 3y = 5e^{2x} \rightarrow y = c_1 e^{-x} + c_2 e^{-3x} + \frac{1}{3} e^{2x}$$

Ex. $y'' + 4y' + 3y = x$

$$D^2x = 0 \rightarrow D^2(y'' + 4y' + 3y) = 0, \quad y = c_1e^{-x} + c_2e^{-3x} + c_3 + c_4x \rightarrow 4c_4 + 3(c_3 + c_4x) = x$$

$$c_4 = \frac{1}{3}, \quad c_3 = -\frac{4}{9}, \quad y = c_1e^{-x} + c_2e^{-3x} + \frac{1}{3}x - \frac{4}{9}$$

Ex. $y'' + 3y' + 2y = 10e^{3x} + 4x^2$

Ex. $y'' + 5y' + 6y = 3e^{3x} + 2e^{-2x}$

$$y_h = c_1e^{-2x} + c_2e^{-3x}, \quad y_p = \frac{1}{10}e^{3x} + 2xe^{-2x}, \quad c_3 = 2$$

Ex. $y'' + y = 3\sin x$

Method 1

$$y_h = c_1 \sin x + c_2 \cos x \rightarrow y = u_1(x) \sin x + u_2(x) \cos x$$

Let $u_1'(x) \sin x + u_2'(x) \cos x = 0 \rightarrow y' = u_1(x) \cos x - u_2(x) \sin x$

$$y'' = u_1'(x) \cos x - u_2'(x) \sin x - u_1(x) \sin x - u_2(x) \cos x$$

$$u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x + u_1 \sin x + u_2 \cos x = 3 \sin x$$

$$u_1' \cos x - u_2' \sin x = 3 \sin x, \quad u_1'(x) \sin x + u_2'(x) \cos x = 0$$

$$u_1' = 3 \sin x \cos x = \frac{3}{2} \sin 2x, \quad u_2' = -3 \sin x \sin x = -3 \frac{1 - \cos 2x}{2}$$

$$u_1 = -\frac{3}{4} \cos 2x + c_1, \quad u_2 = -\frac{3}{2} \left(x - \frac{1}{2} \sin 2x\right) + c_2$$

$$y = -\frac{3}{2} x \cos x + \frac{3}{4} \sin x + c_1 \sin x + c_2 \cos x$$

Method 2

$$(D^2 + 1)(D^2 + 1)y = 3(D^2 + 1) \sin x = 0$$

$$y = c_1 \sin x + c_2 \cos x + c_3 x \sin x + c_4 x \cos x$$

$$(D^2 + 1)y = 2c_3 \cos x - 2c_4 \sin x = 3 \sin x, \quad c_3 = 0, \quad c_4 = -\frac{3}{2},$$

$$y = c_1 \sin x + c_2 \cos x - \frac{3}{2} x \cos x$$

Ex. $(D^4 + 8D^2 + 16)y = -\sin x$

Let $y_h = e^{\lambda x} \rightarrow \lambda^4 + 8\lambda^2 + 16 = 0 = (\lambda^2 + 4)^2, \lambda = \pm 2j, \pm 2j$

$$y_h = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x$$

$$y_p = -\frac{1}{1-8+16} \sin x = -\frac{1}{9} \sin x$$

Ex. $y'' - 7y' + 6y = f(x)$

Hint: $y = e^x u(x) + e^{6x} v(x)$

Ex. $y''' - 2y'' - y' + 2y = \cosh 3x$

$$y_h = e^{\lambda x}, \lambda^3 - 2\lambda^2 - \lambda + 2 = 0 = (\lambda - 1)(\lambda - 2)(\lambda + 1), y_h = c_1 e^x + c_2 e^{2x} + c_3 e^{-x},$$

$$\cosh 3x = \frac{e^{3x} + e^{-3x}}{2}, y_p = a_1 e^{3x} + a_2 e^{-3x}$$

$$a_1 = \frac{1}{16}, a_2 = -\frac{1}{80}$$

§. Euler-Cauchy differential Eq.

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_n y = 0,$$

By use $x = e^z$ or $z = \ln x \rightarrow \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \rightarrow x \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{dy}{dz} = D_z y,$

$$\frac{d^2 y}{dx^2} = \frac{d}{dz} \left(\frac{1}{x} \frac{dy}{dz} \right) = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{dz}{dx} = \frac{1}{x^2} \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) \rightarrow x^2 \frac{d^2 y}{dx^2} = D_z (D_z - 1) y,$$

$$x^3 \frac{d^3 y}{dx^3} = D_z (D_z - 1)(D_z - 2) y, \dots x^n \frac{d^n y}{dx^n} = D_z (D_z - 1)(D_z - 2) \dots (D_z - n + 1) y$$

Ex. $x^2 y'' + xy' + 9y = 0$

$$z = \ln x, [D_z (D_z - 1) + D_z + 9]y = 0, [D_z^2 + 9]y = 0, y = c_1 \cos 3z + c_2 \sin 3z,$$

$$y = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)$$

Ex. $x^3 y''' + 4x^2 y'' - 5xy' - 15y = x^4$

$$z = \ln x, [D_z (D_z - 1)(D_z - 2)y + 4D_z (D_z - 1)y - 5D_z y - 15]y = e^{4z}$$

$$[D_z^3 + D_z^2 - 7D_z - 15]y = e^{4z}$$

$$y_h = e^{\lambda z}, \quad \lambda^3 + \lambda^2 - 7\lambda - 15 = 0 = (\lambda - 3)(\lambda^2 + 4\lambda + 5), \quad \lambda = 3, \quad \lambda = -2 \mp j$$

$$y_h = c_1 e^{3z} + e^{-2z}(c_2 \cos z + c_3 \sin z), \quad y_p = \frac{1}{27} e^{4z}$$

$$y = c_1 e^{3z} + e^{-2z}(c_2 \cos z + c_3 \sin z) + \frac{1}{27} e^{4z}$$

$$y = c_1 x^3 + x^{-2}[c_2 \cos(\ln x) + c_3 \sin(\ln x)] + \frac{1}{27} x^4$$

$$\text{Ex. } a_0(Ax+B)^n y^{(n)} + a_1(Ax+B)^{n-1} y^{(n-1)} + \dots + a_0 y = f(Ax+B)$$

$$z = \ln(Ax+B), \quad \frac{dy}{dx} = \frac{A}{Ax+B} \frac{dy}{dz} \rightarrow (Ax+B) \frac{dy}{dx} = A D_z y$$

$$(Ax+B)^2 \frac{d^2 y}{dx^2} = A^2 D_z(D_z-1)y, \dots, \quad (Ax+B)^n \frac{d^n y}{dx^n} = A^n D_z(D_z-1)\dots(D_z-1+n)y$$

$$\text{Ex. } (x-2)^2 y'' + 2(x-2)y' - 6y = 0$$

$$z = \ln(x-2), \quad [D_z(D_z-1) + 2D_z - 6]y = 0 \rightarrow [D_z^2 + D_z - 6]y = 0$$

$$y = c_1 e^{-3z} + c_2 e^{2z} = c_1 (x-2)^{-3} + c_2 (x-2)^2$$

HW.

$$1. \text{ a) } \frac{dy}{dx} = \frac{2x-y}{x-2y}, \text{ b) } \frac{dy}{dx} = \frac{2x-y+4}{x-2y+5}, \text{ c) } \frac{dy}{dx} = \frac{2x-y+4}{4x-2y+6},$$

$$\text{d) } (x-y)dx + xdy = 0$$

$$2. \quad y^2 dx + (3xy - 4y^3)dy = 0 \quad \text{Hint: consider } x \text{ as the dependent variable.}$$

$$3. \quad y''' + y'/(x-1) = x-1 \quad \text{Hint: consider } y' \text{ as the dependent variable.}$$

$$4. \quad 3xydx + (2y^2 - x^2)dy = 0 \quad \text{Hint: consider } x \text{ as dependent on } y.$$

$$5. \quad 3x^2 dx + 2xydy + y^2 dx = 0$$

$$6. \text{ Let } u = x \text{ be one solution of } y' = x^3(y-x)^2 + y/x$$

7. $x^2 y'' - 3xy' + 3y = 0$

8. $x^3 y''' - 3x^2 y'' + x(6 - x^2)y' - (6 - x^2)y = 0$

9. a) $y'' + y = \sin x$, b) $y'' + 2y' + y = e^{-x} / x$, c) $y'' + 4y' + 4y = e^{-2x} / x^2$

10. a) $y'' - 3y' + 2y = -e^{-2x} / (1 + e^x)$, b) $y'' + 2y' + 2y = 2e^{-x} \tan^2 x$

11. a) $y^{iv} + 3y'' - 4y = \sinh x - \sin^2 x$, b) $y'' + 2ay' + (a^2 - b^2)y = f(x)$

12. $x^2 y'' - 6y = 1 + \ln x$

13. $x^4 y^{iv} + 6x^3 y''' + 15x^2 y'' + 9xy' - 9y = 3 \ln x + 1 / x^2$

14. a) Is $y = x^2$ one solution of $x^2 y'' + x^3 y' - 2(1 + x^2)y = 0$?

b) If $y = x^2$ is one solution, then what is the complete solution?

15. $9y'' + 16y = 32 - 60e^x + 40 \cos 2x$

16. a) Is $y = x$ one homogeneous solution,

b) Find the complete solution.

17. $y'' + 5y' + 6y = 3e^{-2x} + e^{3x}$

18. $x^3 y''' + 2x^2 y'' + xy' - y = 15 \cos(2 \ln x)$